ON "PUMPING TRANSFER OF ENERGY" BETWEEN NONLINEARLY COUPLED OSCILLATORS IN THIRD-ORDER RESONANCE

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The problem of oscillations of two nonlinearly coupled oscillators in resonance seems to have been considered for the first time by the authors of [1], who studied the plane oscillations of an elastic pendulum (a point on a spring) (*).

For small oscillations this system can be regarded as two nonlinearly coupled subsystems (two oscillators) (see Sect. 3 for details). The authors of [1] investigated the case of (2 : 1) resonance of the "vertical" and "horizontal" oscillation frequencies of an elastic pendulum.

They employed one of the variants of perturbation theory, namely the "method of secular perturbations" [2]. In this method the variables are separated into "rapidly" and "slowly" varying ones, and averaging is carried out over the rapidly varying variables. The application of this method to the resonance case has not been sufficiently justified.

We note that some advances in the study of resonance cases in Hamiltonian systems (see [3]) have been made recently.

The reduction of nonlinear Hamiltonian systems in the resonance case to the so-called "normal form" [3] (which is in a certain sense the simplest form) makes it possible to advance the study of nonlinear systems by considering their normal forms. This approach has already yielded some results on the stability of Hamiltonian systems in resonance [3-7]. In the present paper reduction to the normal form is applied to the study of the oscillations of the Hamiltonian system describing nonlinearly coupled oscillators in the case of third-order resonance.

1. Statement of the problem. Let us consider the Hamiltonian system describing n nonlinearly coupled oscillators, i.e. let us assume that the Hamiltonian of the system is of the form

$$H(p, q) = H_2(p, q) + H_3(p, q) + \ldots + H_i(p, q) + \ldots \tag{1.1}$$

where $H_i(p, q)$ are homogeneous polynomials of degree i; here

$$H_2(p, q) = \sum_{i=1}^{n} \beta_i (p_i^2 + q_i^2), \quad \beta_i > 0, \quad \begin{pmatrix} p = (p_1, \ldots, p_n) \\ q = (q_1, \ldots, q_n) \end{pmatrix} \tag{1.2}$$

In the above expression $\pm i\beta_v$ are the eigenvalues of the linearized system (**).

We also assume that there are no multiple eigenvalues, i.e. that $\beta_i \neq \beta_j$ if $i \neq j$. Let the relation

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*) It is interesting to note that this model problem arose in connection with the investigation of oscillations in the CO_2 (carbonic acid) molecule and that the qualitative results provided by this model explain the "splitting of Raman scattering lines in carbonic acid" [1].

**) It is clear that $H_2(p, q)$ can always be reduced to this form, provided it is positive-definite.

$$k_1\beta_1 + k_2\beta_2 + \ldots + k_n\beta_n = 0 \tag{1.3}$$

where k_i are integers, be satisfied. From now on we assume the existence of just one (i.e. to within a constant factor) relation of the form (1.3).

The vector $\mathbf{k} = (k_1, \ldots, k_n)$ is called the resonance vector. The number $k = |k_1| + \ldots + |k_n|$ is called the order of resonance.

In the event of resonance, i.e. of a relation of the form (1.3), system (1.1) is reducible to the so-called "normal form" [3], which is in a certain sense the simplest form.

Let us consider the set of integer vectors k for which (1.3) is fulfilled. We denote the linear shell of these vectors by L. Let us consider the Hamiltonian $\Gamma(\xi, \eta)$. We introduce the complex variables $\zeta_\nu = \xi_\nu + i\eta_\nu, \quad \bar{\zeta}_\nu = \xi_\nu - i\eta_\nu \quad (\nu = 1, \ldots, n)$

and expand $\Gamma(\xi, \eta)$ in a series in ζ_ν, $\bar{\zeta}_\nu$. The general term of this series is of the form

$$\zeta^a\bar{\zeta}^b = \prod_{\nu=1}^{n} \zeta_\nu^{a_\nu}\bar{\zeta}_\nu^{b_\nu}$$

where $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ are integer vectors.

We say that the Hamiltonian Γ is in normal form if its series expansion contains only the "resonance" terms $\zeta^a\bar{\zeta}^b$, where $a - b \in L$.

The possibility of reducing Hamiltonian (1.1) to normal form is guaranteed by the theorem (e.g. see [3, 6]) whereby there exists a canonical transformation $(q, p \to \xi, \eta)$ such that Hamiltonian (1.1) is transformed into the Hamiltonian $\dot{\Gamma}(\xi, \eta)$ in normal form.

From now on we shall confine our attention to the behavior of the system in the lowest order of resonance which satisfies (1.3). The following variant of the indicated theorem will prove useful [6]:

Theorem 1.1. Let the lowest order of resonance defined by relation (1.3) be m. Then there exists a real polynomial canonical substitution of variables $(q, p \to \xi, \eta)$ of degree $m - 1$ such that Hamiltonian (1.1) becomes the Hamiltonian

$$\Gamma^* = \sum_{\nu=1}^{n} \beta_\nu\rho_\nu + \ldots + H_i(\rho) + \ldots + H_\mu(\rho) + \Gamma_m(\rho, \psi) + R(\rho, \varphi)$$

$$\rho = (\rho_1, \ldots, \rho_n) \quad \varphi = (\varphi_1, \ldots, \varphi_n) \quad \psi = \sum_{\alpha=1}^{n} k_\alpha\varphi_\alpha, \quad \mu = \left[\frac{m}{2} - 1\right]$$

Here ρ_α, φ_α are the canonical polar coordinates defined by the relations

$$\xi_\alpha = \sqrt{\rho_\alpha}\sin\varphi_\alpha, \qquad \eta_\alpha = \sqrt{\rho_\alpha}\cos\varphi_\alpha \qquad (\alpha = 1, \ldots, n)$$

ψ is the "resonance phase", $H_i(\rho)$ is a homogeneous polynomial of degree i in the variables ρ_α.

Further,
$$\Gamma_m(\rho, \psi) = \begin{cases} 2A\sqrt{\rho^{|k|}}\cos\psi & \text{if } m = 2d+1, \ d \geqslant 1 \\ 2A\sqrt{\rho^{|k|}}\cos\psi + A_l\rho^{|l|}, & \text{if } m = 2d, \quad d \geqslant 2 \end{cases}$$

$$k = \sum_{\alpha=1}^{n} |k_\alpha| = m, \qquad l = \sum_{\alpha=1}^{n} |l_\alpha| = d, \qquad \rho^{|k|} = \rho_1^{|k_1|}\rho_2^{|k_2|}\ldots\rho_n^{|k_n|}$$

$$A_l\rho^{|l|} = \sum_{l=d} A_{l_1l_2\ldots l_n}\rho_1^{|l_1|}\rho_2^{|l_2|}\ldots\rho_n^{|l_n|}$$

Here $R(\rho, \varphi)$ is of degree higher than $m/2$ in the variables ρ_α; $l = (l_1, \ldots, l_n)$ is an integer vector.

The Hamiltonian

$$\Gamma = \Gamma^* - R(\rho, \varphi) \tag{1.4}$$

coincides with the normal form of the Hamiltonian Γ^* to within terms of order higher than $m/2$ in the variables ρ_α.

The system described by Hamiltonian (1.4) is called an " m-model system". From now on we shall consider three-model systems only. (In the case of third-order resonances this means that terms of order higher than $^3/_2$ in the variables ρ_α have been discarded from the normal form of the Hamiltonian).

We know [3, 6] that

$$J_\alpha = \rho_\alpha - \frac{k_\alpha}{k_1}\rho_1 \quad (\alpha = 2, \ldots, n), \qquad F = \Gamma - \sum_{\nu=1}^{n} \beta_\nu \rho_\nu \tag{1.5}$$

are the independent integrals of the system with Hamiltonian (1.4).

Here $k_1, \ldots, k_n$ are the components of the resonance vector $\mathbf{k}$ ($k_1 \neq 0$).

We note that Hamiltonian essentially depends on $(n + 1)$ variables, namely on ρ_i ($i = 1, \ldots, n$) and on the resonance phase ψ. The system with Hamiltonian (1.4) is integrable provided integrals (1.5) exist (the Liouville theorem).

Qualitative investigation of the behavior of system (1.4) in accordance with the initial conditions can be conveniently carried out by using integrals (1.5) to eliminate n variables and to obtain a first-order autonomous differential equation for one (any) of the variables ρ_i. Once one ρ_i has been determined, the rest can be found from the integrals J_α ; the phases φ_i can then be determined by quadratures from the corresponding equations for φ_i defined by Hamiltonian (1.4).

In some cases it is sufficient to investigate the behavior of the variables ρ_i alone, since, as will be shown below, they represent the energy of the ith oscillator in the first approximation.

2. Third-order resonance. Third-order resonances correspond to one of the following relationships (with the oscillators numbered accordingly):

$$\beta_1 = 2\beta_2 \quad \text{or} \quad \beta_1 + \beta_2 = \beta_3.$$

We shall consider the resonance $\beta_1 = 2\beta_2$. The authors of [1] investigated this resonance for a system with two degrees of freedom (an elastic pendulum), i.e. for a Hamiltonian of the special form (3.2).

We can show that in the general case of Hamiltonian (1.1), (1.2) (if the Hamiltonian does not contain terms of degree higher than the third in the variables p and q) the qualitative picture of motion for the resonance in question is of the same character as in the problem investigated in [1] (*).

For simplicity (see Note 2.1) we consider a system with two degrees of freedom. In this case the three-model system, i.e. the system defined by a Hamiltonian of the form (1.4) ($m = 3$) is given by the expression

$$\Gamma(\rho, \psi) = \beta_2 (2\rho_1 + \rho_2) + 2A\sqrt{\rho_1 \rho_2^2} \cos\psi, \quad \psi = \varphi_1 - 2\varphi_2 \tag{2.1}$$

We assume that the constant $A \neq 0$.

By virtue of (1.5) system (2.1) has the following integrals:

*) We note that this result partly answers one of the questions posed by M. G. Krein at the Fifth International Conference on Nonlinear Oscillations (Kiev, 1969).

$$J = 2\rho_1 + \rho_2, \qquad F = 2A\sqrt{\rho_1\rho_2^2}\cos\psi \tag{2.2}$$

Integrals (2.2) enable us readily to reduce the equation for ρ_2 to the form

$$\frac{d\rho_2}{dt} = \pm 2A\sqrt{2\rho_2^2(J-\rho_2) - F_1^2}, \qquad F_1 = F/A \tag{2.3}$$

A similar equation, though in somewhat different variables, was investigated in [1].

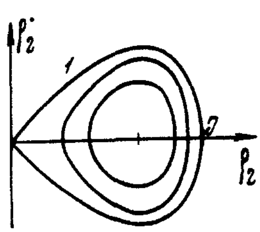

Fig. 1

Figure 1 shows the "phase portrait" (the integral curves) of Eq. (2.3) for several values of the integral F and a fixed value of J.

This phase portrait contains two singular points: a saddle ($\rho_2^{\cdot} = \rho_2 = 0$) and a center ($\rho_2^{\cdot} = 0$, $\rho_2 = {}^2/_3 J$). All of the integral curves which do not pass through these singular points are closed curves (cycles) intersecting the ρ_2-axis at a right angle. In fact

$$\frac{d(\rho_2^{\cdot})}{d\rho_2} = \pm\frac{2A(2J-3\rho_2)\rho_2}{\sqrt{2\rho_2^2(J-\rho_2) - F_1^2}}$$

becomes ∞ at $\rho_2^{\cdot} = 0$ (i.e. when the denominator is equal to zero) provided that $\rho_2(2J - 3\rho_2) \neq 0$.

The center corresponds to motions such that ρ_2, and therefore (see (2.2)) ρ_1, remains constant. For example, in the elastic pendulum problem [1] this corresponds to certain periodic motions of the oscillators. Since the ρ_i represent the energies of the ith oscillators in the first approximation (see the expressions for ρ_i in terms of the initial variables of the elastic pendulum problem, namely expressions (3.8) below), it follows that the figure can be conveniently interpreted in terms of "pumping transfer of energy" [1].

Periodic "pumping transfer of energy" from one oscillator to the other occurs for all possible values of ρ_2 except the values associated with the singular points. This "pumping transfer" proceeds with the period

$$\tau = \oint \frac{d\rho_2}{2A\sqrt{2\rho_2^2(J-\rho_2) - F_1^2}} \tag{2.4}$$

where the integral is taken along the corresponding cycle (these periods can be expressed in terms of elliptic functions).

The saddle ($\rho_2^{\cdot} = \rho_2 = 0$) is also associated with a periodic solution, but is of no interest.

When the initial conditions correspond to a separatrix, what we have is a limiting motion. The representing point arrives at the origin after an infinite time (integral (2.4) diverges). For initial values close to those corresponding to the separatrix we have almost complete "pumping transfer of energy" from one oscillator to the other, and the process lasts "a very long time" [1].

Thus, the picture of the "pumping transfer of energy" described in [1] also applies in the general case of resonance $\beta_1 = 2\beta_2$.

Note 2.1. A similar picture of the "pumping transfer of energy" between resonating oscillators in the case of the above resonance likewise applies when the oscillators are part of the system of n nonlinearly coupled oscillators. In these cases the expressions for ρ_1 and ρ_2 (let the first and second oscillators be in resonance) depend on the variables associated with the remaining oscillators.

3. The Vitt-Gorelik problem (oscillation of an elastic pendu-

lum in resonance (2,1)). Let us consider the problem of oscillation of an elastic pendulum in the case of (2 : 1) resonance of the vertical and horizontal oscillations using the theorem on reduction to the normal form.

Following the authors of [1], we shall consider an elastic pendulum, i. e. a load suspended from a spring; the upper end of the latter is fixed. We assume that the motion occurs in a particular vertical plane. In our expressions r is the instantaneous length of the spring, l_0 the length of the unloaded spring, θ the angle of deviation from the vertical (which we assume to be small at all times), m the mass of the load, k the spring constant, and g the gravitational acceleration. The kinetic and potential energies of the system are given by

$$T = \frac{m}{2}(r'^2 + r^2\theta'^2), \qquad V = \frac{k}{2}(r - l_0)^2 - mgr\left(1 - \frac{\theta^2}{2}\right) + \cdots$$

Let us replace r by the coordinate x equal to the elongation of the spring relative to its static length $l = l_0 + mg / k$, i. e. let us set $x = (r - l) / l$.

Since we are concerned with small oscillations, we can assume that x is very small compared with unity. Neglecting terms of order higher than the third in the small quantities x and θ and their derivatives, we obtain the following new expressions for the kinetic and potential energies:

$$T = \frac{ml^2}{2}(x'^2 + \theta'^2 + 2x\theta'^2), \quad V = \frac{ml^2}{2}\left(\frac{k}{m}x^2 + \frac{g}{l}\theta^2 + \frac{g}{l}x\theta^2\right)$$

Let us introduce the impulses $p_x = \partial T / \partial x', p_\theta = \partial T / \partial \theta'$ associated with the coordinates x and θ. This enables us to express the Hamiltonian as

$$H = T + V = \frac{1}{2m'}(p_x^2 + p_\theta^2) + \frac{m'}{2}(\alpha^2x^2 + \beta^2\theta^2) - \frac{1}{m'}xp_\theta^2 + \frac{m'}{2}\beta^2x\theta^2$$

$$m' = ml^2, \quad \alpha^2 = k / m, \quad \beta^2 = g / l, \quad \alpha, \beta > 0$$

Here α and β are the frequencies of the linear oscillators (in the absence of nonlinear coupling). The linear canonical substitution of variables

$$p_1 = \frac{p_x}{\sqrt{m'\alpha}}, \quad z_1 = \sqrt{m'\alpha}\,x, \quad p_2 = \frac{p_\theta}{\sqrt{m'\beta}}, \quad z_2 = \sqrt{m'\beta}\,\theta \tag{3.1}$$

enables us to reduce the Hamiltonian to the form

$$H = {}^1/_2\,[\alpha(p_1^2 + z_1^2) + \beta(p_2^2 + z_2^2)] - \gamma z_1(2p_2^2 - z_2^2), \qquad \gamma = \frac{\beta}{2\sqrt{m'\alpha}} \tag{3.2}$$

For simplicity (without limiting generality) we set $\alpha = 2$, $\beta = 1$. To reduce Hamiltonian (3.2) to normal form, we seek the generating function $W(z, \eta)$ for the canonical transformation of variables

$$z, p \to \xi, \eta, (z = (z_1, z_2), p = (p_1, p_2), \ \xi = (\xi_1, \xi_2), \ \eta = (\eta_1, \eta_2)).$$

Here

$$\xi_i = \frac{\partial W}{\partial \eta_i}, \qquad p_i = \frac{\partial W}{\partial z_i} \qquad (i = 1, 2) \tag{3.3}$$

Since we are limiting ourselves of terms of order not higher than the third in H, we seek the function $W(z, \eta)$ in the form $W = W_2 + W_3$, where W_2, W_3 are second- and third-order homogeneous polynomials, respectively.

We note that H_2 (second-degree terms in H) is already in normal form, so that W_2 can be taken in the form $W_2 = z_1\eta_1 + z_2\eta_2$. This is equivalent to identity transformation ($\xi_i = z_i$, $p_i = \eta_i$ $(i = 1, 2)$). The reduction of $H = H_2 + H_3$ to the normal form $\Gamma = \Gamma_2 + \Gamma_3$ in accordance with the procedure of reduction to normal form [3] involves the solution of a certain partial differential equation. In our case this equation is of the form

$$2\left(\eta_1\frac{\partial W_3}{\partial z_1}-z_1\frac{\partial W_3}{\partial \eta_1}\right)+\left(\eta_2\frac{\partial W_3}{\partial z_2}-z_2\frac{\partial W_3}{\partial \eta_2}\right)=-{}^1/_4\gamma\,(z_1z_2^2-5z_1\eta_2^2-6z_2\eta_1\eta_2)$$

This equation can be solved by the method of undetermined coefficients (we recall that W_3 is a homogeneous third-degree polynomial).

The solution of the above equation is the function

$$W_3={}^1/_4\gamma\,[3z_1z_2\eta_2-\eta_1(z_2^2+\eta_2^2)]$$

Thus,

$$W=W_2+W_3=z_1\eta_1+z_2\eta_2+{}^1/_4\gamma\,[3z_1z_2\eta_2-\eta_1(z_2^2+\eta_2^2)] \tag{3.4}$$

The transformation defined by this function is of the form

$$z_1=\xi_1+{}^1/_4\gamma\,(\xi_2^2+\eta_2^2),\qquad p_1=\eta_1+{}^3/_4\gamma\xi_2\eta_2$$
$$z_2=\xi_2-{}^1/_4\gamma\,(3\xi_1\xi_2-2\eta_1\eta_2),\qquad p_2=\eta_2+{}^1/_4\gamma\,(3\xi_1\eta_2-2\eta_1\xi_2) \tag{3.5}$$

In the new variables the Hamiltonian becomes

$$H={}^1/_2[2(\xi_1^2+\eta_1^2)+(\xi_2^2+\eta_2^2)]+{}^3/_4\gamma\,(2\xi_2\eta_1\eta_2+\xi_1\xi_2^2-\xi_1\eta_2^2) \tag{3.6}$$

Converting to canonical polar coordinates by means of the formulas

$$\xi_i=\sqrt{\rho_i}\sin\varphi_i,\qquad \eta_i=\sqrt{\rho_i}\cos\varphi_i\ (i=1,2)$$

we obtain the Hamiltonian in the familiar form (2.1),

$$\Gamma=(2\rho_1+\rho_2)+{}^3/_2\gamma\sqrt{\rho_1\rho_2^2}\cos(\varphi_1-2\varphi_2) \tag{3.7}$$

Thus, canonical transformation (3.5) reduces the Hamiltonian of the Vitt-Gorelik problem to the normal form standard for third-order resonances in systems with two degrees of freedom (investigated in Sect. 2).

The expressions for the variables

$$\rho_1=z_1^2+p_1^2-{}^1/_2\gamma\,[z_1(z_2^2+p_2^2)+3z_2p_1p_2]$$
$$\rho_2=z_2^2+p_2^2+{}^1/_2\gamma\,[3z_1(z_1z_2-p_2^2)+2p_1p_2(z_1+z_2)] \tag{3.8}$$

indicate that in the first approximation the ρ_i represent the energies of the i-th oscillators; the third-order terms arise through interaction.

Note 3.1. Let the ratio of frequencies of the vertical and horizontal oscillations in the above elastic pendulum problem be (1 : 2) (and not (2 : 1) as in [1]). The character of Hamiltonian (3.2) is such that on reduction to normal form (to within terms of higher than the third order of smallness) the coefficient A in expression (2.1) is equal to zero. Thus normal form (2.1) is degenerate in this case. This means that in contrast to the usual picture of the "pumping transfer of energy" in the case of third-order resonance in a system with two degrees of freedom, the analysis of nonlinear coupling effects in this case requires the retention of higher-order terms in the initial Hamiltonian (and in the normal form). This picture will be considered in more detail in a future paper.

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THE ACTION-ANGLE VARIABLES IN THE EULER-POINSOT PROBLEM

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Use of the action-angle variables (see e. g. [1]) leads, in a number of cases, to considerable simplification when the perturbation method is applied to study the dynamics of perturbed motion, especially when computing the higher order approximations. Below we obtain such variables for the problem of a solid rotating freely about a fixed point (the Euler-Poinsot case).

Free motion of a solid with a fixed point can be described by a system of canonical equations whose Hamiltonian is [2]

$$H = \frac{G^2 - G_\zeta^2}{2}\left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) + \frac{G_\zeta^2}{2C} \tag{1}$$

Here A, B, C are the principal moments of inertia of the body relative to the fixed point, G is the kinetic moment, G_ζ is its projection on the axis corresponding to the moment of inertia C of the associated coordinate system, and ψ, ϑ, φ are the Euler angles (of precession, nutation and self-rotation) defining the position of the body in the fixed coordinate system of which one axis is collinear with the kinetic moment vector. Position of this vector in the initial absolute coordinate system can be defined by the following two quantities: L which is the projection of the kinetic moment on one axis of the initial coordinate system, and the angle h. The quantities $G, G_\zeta, L, \psi, \varphi, h$ form a complete set of canonical variables for the present problem.

Change to the action-angle variables is effected by means of a canonical transformation which transforms the Hamiltonian H into a function of impulses only, and is independent of the angles.

In our problem we can use the triad G, L, I of impulses as the action variables. Here I is the projection of the kinetic moment on an axis of the associated coordinate system, averaged over the characteristic rotation

$$I = \frac{1}{2\pi}\oint G_\zeta\, d\varphi \tag{2}$$